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Can Dirac Observability Apply to Gravitational Systems?

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The problem of what is observable in general relativity is investigated. With the help of Landau's observable space interval, the observational frames for individual observers are established. Within the Ashtekar formulation of general relativity, we argue from the nonvanishing Poisson brackets of the Yang–Mills field and the constraints that Dirac observability does not apply to gravitational systems.

1. INTRODUCTION

In general, in a theory with a *gauge invariance* (in a generalized sense) it is assumed that only gauge-invariant quantities can be measured and/or predicted by the theory. In general relativity, the problem is still far from being trivial and has been debated (Kuchar and DeWitt, 1991). Actually, one may find in the literature substantially contrasting points of view on this issue (Stachel, 1986). These viewpoints can be classified into two: the *nonlocal point of view* and the *local point of view*. Recently, Rovelli (1991) considered in the metric representation a model in which the matter represents the physical reference system. In this paper, we discuss the problem of observables in the vierbein formulation. Based on the Einstein's observable time and space interval, we take the *local point of view* that any measurement in physics is performed in the local flat reference system whose existence is guaranteed by the equivalence principle. In Section 2, we give the observational local flat systems for different observers with the help of Landau's definition of observed space length and argue that the observed quantities should be the projections of Riemann geometric quantities onto the local flat

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system. In Section 3, from the fact that in the Ashtekar formulation of general relativity the Yang–Mills field, which is empirically observable, does not commute with the gravitational first-class constraints, we argue that Dirac observability does not apply to general relativity.

2. THE MEASURING SIGNIFICANCE OF VIERBEIN

It is well known that in the vierbein formalism of general relativity there are two kinds of quantities: Riemannian quantities such as the curvature $R_{\mu\nu\alpha\beta}$ and Lorentzian quantities such as R_{abmn}, which is related to $R_{\mu\nu\alpha\beta}$ as

$$R_{abmn} = e_a^{\mu} e_b^{\nu} e_m^{\alpha} e_n^{\beta} R_{\mu\nu\alpha\beta} \tag{1}$$

Our conventions are as follows:

$$g_{\mu\nu} = \eta_{ab} e_{\mu}^{a} e_{\nu}^{b}, \qquad \eta_{ab} = \mathrm{diag}(1, -1, -1, -1,)$$

A problem then arises: for a given spacetime, which kind of quantities are observable? For instance, when there exists a Maxwell field, which one of $F_{\mu\nu}$ and F_{ab} is measured in the laboratory? This problem is very important because physics, by its nature, studies observables and their relationship.

Since the measuring of space and time is the foundation of all other measurement, we should study first the observable time and spatial interval of two world events x and $x + dx$ whose interval is dx^{μ}. Einstein (1979) pointed out that in a local system, for two neighboring point events,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = \Delta T^2 - \Delta L^2 \tag{2}$$

where ΔL is measured directly by a measuring rod and ΔT by a clock at rest relative to the system: these are the naturally measured lengths and times. Though there is still an ambiguity in this statement, which is that ΔT and ΔL are different in different Lorentz gauges, one point is clear: what is measured is ΔT and ΔL instead of dx^{μ}. According to Hawking (1988), in the theory of relativity, there is no unique absolute time, but instead each individual has his own personal measure of time that depends on where he is and how he is moving. This argument is in good agreement with that of Einstein. Moreover, it indicates that an observer is characterized by his position and velocity, i.e., an observer can be denoted by $\mathcal{O}(x, u)$. With regard to physical measurement, Wald (1984) pointed out that an observer equipped with measuring apparatus can be characterized by an orthonomal basis which forms a locally inertial frame e_{μ}^{a} of which the first denotes the alignment of his time rod and the other three serve as references for how he aligns the measuring apparatus. So according to this argument, the observed quantities are the projections onto the basis e_{μ}^{a} of the quantities in general spacetime. Therefore, the key problem is how to determine the basis.

Let us study first the case for a static observer $\mathcal{O}$ whose velocity is $u^\mu = (g_{00}^{-1/2}, 0, 0, 0)$. The measured time and space interval for a static observer is (Landau and Lifshitz, 1951)

$$\Delta T = \frac{g_{0\mu}}{\sqrt{g_{00}}} dx^\mu \tag{3}$$

$$\Delta L^2 = \left(\frac{g_{0i}g_{0j}}{g_{00}} - g_{ij}\right) dx^i\, dx^j \tag{4}$$

We next show that this is equivalent to saying that the time rod of $\mathcal{O}$ is fixed by

$$e^0_\mu = u_\mu \tag{5}$$

Since

$$u_\mu = g_{\mu\nu}u^\nu = \frac{g_{\mu 0}}{\sqrt{g_{00}}} \tag{6}$$

then if equation (5) holds, from ($a' = 1, 2, 3$)

$$g_{00} = (e^0_0)^2 - (e^{a'}_0)^2 \tag{7}$$

we have

$$e^{a'}_0 = 0 \tag{8}$$

so from equation (1), we have equations (3) and (4). Equations (5) and (7) determine the time rod of $\mathcal{O}$; the directions of the spatial rods remain arbitrary, i.e., the observer may rotate arbitrarily his apparatus and this does not affect his observed results.

For a moving observer $\mathcal{O}'$, how do we fix the basis? As a principle, we generalize equation (5) to the case for arbitrary u^μ. The reason for this generalization is that it makes the observed time and thus the observed length to be coordinates x^μ-independent. Suppose that his velocity is $u'^\mu \neq u^\mu$; denote the basis as $\bar{e}^a_\mu$; then according to special relativity, $\bar{e}^a_\mu$ should be related to e^a_μ by a local Lorentz transformation, which is determined by the relative Lorentz velocity $u'^a = u'^\mu e^a_\mu$:

$$\bar{e}^a_\mu(x) = \Lambda^a{}_b(u'^a)e^b_\mu(x) \tag{9}$$

where $\Lambda^a{}_c\Lambda^b{}_d\eta_{ab} = \eta_{cd}$. Thus Λ is determined up to a spatial rotation,

$$\Lambda^0{}_b = u'_b \tag{10}$$

With this definition of the observable space and time interval, we can solve the problems of the twin paradox and the red shift. Let us discuss the latter. Consider two static observers A and B; two light waves start at x^μ_A and

$x_A^\mu + dx_A^\mu$ to travel from A to B. The world lines for each wave, C_1 and C_2, are null, so

$$dt = \frac{-g_{0i}dx^i + \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}}{g_{00}} \tag{11}$$

In general, equation (11) is a differential equation; in some special cases, for example, the static metric and the Robertson–Walker (RW) metric, it can be readily integrated. After equation (11) is solved, we can obtain the world points x_B^μ and $x_B^\mu + dx_B^\mu$ at which the two waves arrive at B. Then the red shift is

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{g_{00}(x_B)}{g_{00}(x_A)}}\,\frac{g_{0\mu}(x_A)dx_A^\mu}{g_{0\mu}(x_B)dx_B^\mu} = \sqrt{\frac{g_{00}(x_A)}{g_{00}(x_B)}}\,\frac{dx_A^0}{dx_B^0} \tag{12}$$

For a static metric such as the Schwarzschild metric, it can be easily shown that $dx_A^0 = dx_B^0$, so we have

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{g_{00}(x_A)}{g_{00}(x_B)}} \tag{13}$$

For the time-dependent RW metric, we have

$$\frac{dx_A^0}{dx_B^0} = \frac{R(x_A^0)}{R(x_B^0)} \tag{14}$$

so the red shift is given by

$$\frac{\nu_B}{\nu_A} = \frac{R(x_A^0)}{R(x_B^0)} \tag{15}$$

Though the argument that the quantity $F_{\mu\nu}$ is not observable may sound strange, it can be illustrated by the following example. Suppose that there is a uniform electric field $\mathbf{E} = (0, 0, E)$ in the Minkowski spacetime. An observer S rotates around the z axis with angular velocity ω; the distance between S and the z axis is r, $r\omega \leq c$. Denote the coordinates of the static frame as $X^\mu = (cT, R, \Theta, Z)$, those of the rotating frame as $x^\mu = (ct, r, \theta, z)$. Then we have

$$R = r, \qquad Z = z, \qquad T = t, \qquad \theta = \Theta + \omega t \tag{16}$$

Hence

$$\begin{aligned} ds^2 &= c^2dT^2 - dR^2 - R^2d\Theta^2 - dZ^2 = G_{\mu\nu}dX^\mu dX^\nu \\ &= \left(1 - \frac{\omega^2r^2}{c^2}\right)c^2dt^2 - dr^2 - r^2d\theta^2 - dz^2 + \frac{2\omega r^2}{c}\,d\theta d(ct) = g_{\mu\nu}dx^\mu dx^\nu \end{aligned} \tag{17}$$

and so to S, the spacetime is no longer Minkowski. According to the discussion above, the observation frame of S is given by

$$e^a_\mu(x) = \begin{pmatrix} \sqrt{1-\omega^2 r^2/c^2} & 0 & \omega r^2/\sqrt{c^2-\omega^2 r^2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & rc/\sqrt{c^2-\omega^2 r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

In the static frame, the electromagnetic tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E/c & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

Hence the electromagnetic tensor in the rotating frame is given by

$$f^{\mu\nu} = \frac{\partial x^\mu}{\partial X^\alpha}\frac{\partial x^\nu}{\partial X^\beta} F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega E/c^2 \\ E/c & 0 & \omega E/c^2 & 0 \end{pmatrix} \tag{20}$$

i.e.,

$$f^{ab} = \begin{pmatrix} 0 & 0 & 0 & -\gamma E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma\omega r E/c^2 \\ \gamma E/c & 0 & \gamma\omega r E/c^2 & 0 \end{pmatrix} \tag{21}$$

where $\gamma = 1/\sqrt{1 - r^2\omega^2/c^2}$. That is, the electromagnetic field S observed is

$$\mathbf{E} = (0, 0, \gamma E), \qquad \mathbf{B} = (\gamma\omega r E/c^2, 0, 0) \tag{22}$$

According to special relatively, the relationship of electromagnetic fields in two inertial frames is given by

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel \tag{23}$$

$$\mathbf{E}'_\perp = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_\perp, \qquad \mathbf{B}'_\perp = \gamma\left(\mathbf{B} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}\right)_\perp \tag{24}$$

When $v \to c$, $\mathbf{E}_\perp \to \infty$, $\mathbf{B}_\perp \to \infty$; hence it is obvious that $f^{\mu\nu}$ is not the observed quantity, whereas f^{ab} is.

We make some remarks. First, according to equation (1), the observed speed of light is always constant and isotropic. Second, it is necessary to introduce the vierbein formalism not only for mathematical reasons, but also because of physical measurement considerations. Third, the general covariant conservative energy-momentum and angular momentum obtained in Duan *et al.* (1988), Duan and Feng (1996), Feng and Zong (1996), Feng and Duan (1995), and Feng (1996) are actually observable.

3. LANDAU'S SPACE INTERVAL AND YANG–MILLS FIELD DO NOT COMMUTE WITH THE CONSTRAINTS

In the Ashtekar formalism of gravity coupled to the Yang–Mills field, the Lagrangian is (the indices μ, ν, etc., go from 0 to 3) (Ashtekar *et al.*, 1989)

$$\mathcal{L} = ({}^4\sigma)\sigma^{\mu}{}_{A}{}^{A'}\sigma^{\nu}{}_{BA'}\,{}^4F_{\mu\nu}{}^{AB} + \tfrac{1}{2}({}^4\sigma)g^{\mu\nu}g^{\alpha\beta}\,\mathrm{Tr}\,{}^4\mathbf{F}_{\mu\alpha}\,{}^4\mathbf{F}_{\nu\beta} \tag{25}$$

After 3 + 1 decomposition, four constraints can be obtained,

$$\tilde{C}_{AB} = i\sqrt{2}\mathcal{D}_{\mu}\tilde{\sigma}^{\mu}_{AB} \approx 0 \tag{26}$$

$$\tilde{C}_{\mu} = i\sqrt{2}\,\mathrm{Tr}(\tilde{\sigma}^{\nu}F_{\mu\nu}) - \mathrm{Tr}(\tilde{\mathbf{E}}^{\nu}\mathbf{F}_{\mu\nu}) \approx 0 \tag{27}$$

$$\tilde{\tilde{C}} = -\mathrm{Tr}(\tilde{\sigma}^{\mu}\tilde{\sigma}^{\nu}F_{\mu\nu}) + \tfrac{1}{8}\sigma^{-2}\,\mathrm{Tr}(\tilde{\sigma}^{\mu}\tilde{\sigma}^{\nu})\,\mathrm{Tr}(\tilde{\sigma}^{\alpha}\tilde{\sigma}^{\beta})\,\mathrm{Tr}(\mathbf{E}_{\mu\alpha}\mathbf{E}_{\nu\beta} + \mathbf{B}_{\mu\alpha}\mathbf{B}_{\nu\beta}) \approx 0 \tag{28}$$

$$\tilde{\mathbf{C}} = \mathbf{D}_{\mu}\tilde{\mathbf{E}}^{\mu} \approx 0 \tag{29}$$

where the boldface letters refer to the Yang–Mills field, and Tr denotes the trace over the gravitational $SU(2)$ spin indices and the trace over the Yang–Mills internal indices. $\tilde{\mathbf{E}}^{\mu}$ (the Yang–Mills electric field) is the canonical momentum of the field $\mathbf{A}_{\mu}$, which is the projection of the Yang–Mills field ${}^4\mathbf{A}_{\mu}$ onto the Cauchy surface Σ_t; $\mathbf{E}_{\mu\nu}$, $\mathbf{B}_{\mu\nu}$ are dual to the Yang–Mills electric and magnetic fields, respectively.

Denote the generators of the Yang–Mills internal group as λ_I, $\mathbf{A}_{\mu} = \mathbf{A}^I_{\mu}\lambda_I$, $[\lambda_I, \lambda_J] = f_{IJK}\lambda_K$. In (29), $\mathbf{D}_{\mu}$ is the Yang–Mills connection determined by $\mathbf{A}_{\mu}$. The fundamental Poisson brackets are

$$\{\tilde{\sigma}^{\mu}_{AB}(x), A^{MN}_{\nu}(y)\} = -\frac{i}{\sqrt{2}}\,\delta(x-y)\delta^{\mu}_{\nu}\delta^{M}_{(A}\delta^{N}_{B)} \tag{30}$$

$$\{\tilde{\mathbf{E}}^{\mu}_{I}(x), \mathbf{A}^{J}_{\nu}(y)\} = \delta(x-y)\delta^{\mu}_{\nu}\delta^{J}_{I} \tag{31}$$

Using the smeared constraint functionals

$$C_{\underline{N}} = i\sqrt{2}\int_{\Sigma} d^3x\,\underline{N}\tilde{\tilde{C}} \tag{32}$$

$$C_{\vec{N}} = -\int_{\Sigma} d^3x\,N^{\mu}\tilde{C}_{\mu} \tag{33}$$

$$C_{N,\mathbf{N}} = \int_\Sigma d^3x\, \mathrm{Tr}(N\tilde{C} + \mathbf{N}\tilde{\mathbf{C}}) \tag{34}$$

$$\mathbf{C}_{\overline{N}} = C_{\overline{N}} - \int_\Sigma d^3x\, \mathrm{Tr}[N^\mu(A_\mu\tilde{C} + \mathbf{A}_\mu\tilde{\mathbf{C}})] \tag{35}$$

we have the Poisson brackets

$$\{C_{N,\mathbf{N}}, \tilde{\mathbf{E}}^\mu\} = [\mathbf{N}, \tilde{\mathbf{E}}^\mu], \qquad \{C_{N,\mathbf{N}}, \mathbf{A}_\mu\} = -\mathbf{D}_\mu\mathbf{N} \tag{36}$$

$$\{\mathbf{C}_{\overline{N}}, \tilde{\mathbf{E}}^\mu\} = \mathcal{L}_{\overline{N}}\tilde{\mathbf{E}}^\mu, \qquad \{\mathbf{C}_{\overline{N}}, \mathbf{A}_\mu\} = \mathcal{L}_{\overline{N}}\mathbf{A}_\mu \tag{37}$$

$$\{C_{\underline{N}}, \tilde{\mathbf{E}}^\lambda_J(x)\} = -\frac{i}{\sqrt{2}}\,\mathrm{Tr}[\mathbf{D}_\beta(\underline{N}\sigma^{-2}\,\mathrm{Tr}(\tilde{\sigma}^\mu\tilde{\sigma}^\lambda)\,\mathrm{Tr}(\tilde{\sigma}^\nu\tilde{\sigma}^\beta)\mathbf{F}_{\mu\nu})\lambda_J] \tag{38}$$

$$\{C_{\underline{N}}, \mathbf{A}^I_\alpha(x)\} = \frac{i}{2\sqrt{2}}\,\underline{N}\sigma^{-2}\,\mathrm{Tr}(\tilde{\sigma}^\mu\tilde{\sigma}^\nu)\,\mathrm{Tr}(\tilde{\sigma}^\lambda\tilde{\sigma}^\beta)\underline{\eta}_{\mu\alpha\lambda}\underline{\eta}_{\nu\beta\tau}\tilde{\mathbf{E}}^\tau_L\,\mathrm{Tr}\,\lambda_I\lambda_L \tag{39}$$

Since

$$\{C_{\underline{N}}, \tilde{\sigma}^\mu\} = -2\mathcal{D}_\nu(\underline{N}\tilde{\sigma}^{[\mu}\tilde{\sigma}^{\nu]}) \tag{40}$$

$$\{C_{N,\mathbf{N}}, \tilde{\sigma}^\mu\} = [N, \tilde{\sigma}^\mu] \tag{41}$$

$$\{\mathbf{C}_{\overline{N}}, \tilde{\sigma}^\mu\} = \mathcal{L}_{\overline{N}}\tilde{\sigma}^\mu \tag{42}$$

we have

$$\{C_i, \tilde{\mathbf{E}}_{I,AB}\} \neq 0, \qquad \{C_i, \mathbf{A}^I_{AB}\} \neq 0 \tag{43}$$

where $C_i = C_{N,\mathbf{N}}, \mathbf{C}_{\overline{N}}, C_{\underline{N}}$ and $\tilde{\mathbf{E}}_{I,AB} = \sigma_{\mu AB}\tilde{\mathbf{E}}^\mu_I$, $\mathbf{A}^I_{AB} = \sigma^\mu_{AB}\mathbf{A}^I_\mu$, which are observed quantities according to the discussion of the last section.

From the 3 + 1 decomposition of the metric

$$g_{00} = h_{\mu\nu}N^\mu N^\nu - N^2, \qquad g_{0\mu} = h_{\mu\nu}N^\nu \tag{44}$$

$$g_{\mu\nu} = h_{\mu\nu} = -\mathrm{Tr}\,\sigma_\mu\sigma_\nu \qquad (\mu = 1, 2, 3) \tag{45}$$

it is easy to obtain that

$$\{C_i, h_{\mu\nu}\} = \frac{\delta(\sigma_{\mu MN}\sigma_\nu{}^{MN})}{\delta\tilde{\sigma}^{\lambda AB}}\{C_i, \tilde{\sigma}^{\lambda AB}\} \tag{46}$$

$$\{C_i, dL^2\} = \{C_i, h_{\mu\nu}\}\,dx^\mu\,dx^\nu - \left\{C_i, \frac{g_{0\mu}g_{0\nu}}{g_{00}}\right\}dx^\mu\,dx^\nu \tag{47}$$

Hence, in general,

$$\{C_i, dL^2\} \neq 0 \tag{48}$$

Similarly, the bracket of C_i and dT is not zero in general,

$$\{C_i, dT\} \neq 0 \tag{49}$$

Since the Yang–Mills field and the space interval are empirically observable, equations (43), (48), and (49) imply that physical observables may not commute with the first-class constraints in curved spacetime.

We make some final comments. In constrained systems, it is generally accepted that physical observables should commute with the first-class constraints which generate *gauge* transformations in the general sense (Dirac, 1964; Gitman and Tyutin, 1990). From (43), (48), and (49), it can be concluded that in general relativity, we cannot follow this dogma. Similar situations occur in the parametrized models (Huang *et al.*, n.d.). How are we to understand the difference? As is discussed in Section 2, the measuring system of an observer is determined by his position and velocity, i.e., the measured results depend on the specific spacetime position and the local Lorentz system in the specific gauge. The constraints (26)–(29) generate changes in both the spacetime position and the local Lorentz gauge, and thus may generate changes in the measured results; hence the observable quantities need not commute with them. But in other systems, such as QED and QCD, the Yang–Mills constraints generate only internal gauge transforms rather than changes in the position and local Lorentz gauge, and so do not affect the measured results; hence it is natural to require that the observables should commute with the first-class constraints.

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